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Decentralized Network Interdiction Games

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Decentralized Network Interdiction Games
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Abstract

This project established the theoretical and computational foundation for a new class of games, termed as the multi-interdictor network games (MINGs). Such games naturally arise in the context of various military and security applications in which multiple interdictors with differing objectives operating on a common network. Within the general class of MINGs, we proved existence of Nash equilibria for specific subclasses of such games, including the shortest-path and maximum-flow multi-interdictor network games, and developed scalable algorithms to compute such equilibria. In addition, we provided theoretical bounds on the worst-case efficiency loss of equilibria for the shortest-path games, with such loss caused by the lack of coordination among noncooperative interdictors, and prosed an empirical approach utilizing decentralized algorithms to study the average-case efficiency loss. Finally, we developed convergent decentralized algorithms based on the connection between the class of potential games (to which the MINGs belong) and optimization that will advance the computation of large-scale optimization problems.

Contents

1 Introduction and Objectives

This research project is mainly motivated by the game-theoretic aspect of multi-interdictor games. Interdiction problems, in which defenders seek to destroy, neutralize, or delay their enemy's potential to launch effective attacks, naturally arise in a variety of military and homeland security contexts, such as coordinating tactical air strikes, combatting drug trafficking, protecting civil infrastructure against terrorist attacks, defending against the smuggling of nuclear material, and preventing attacks on critical computer communications networks. Such problems have traditionally been looked at from a centralized decision-maker's point of view; that is, it is typically assumed that there is one agent that determines and implements an interdiction strategy in a system. However, in reality, interdiction strategies are often carried out in a decentralized or uncoordinated fashion, especially when there are multiple interdicting agencies. One of the most prominent examples may be the ongoing fight against the terrorist group of ISIS (aka ISIL or Daesh). In such a fight, multiple interdictors with different objectives exist, including the US-led coalition, Russian, Iran, and Saudi Arabia-led Arab countries. While each interdictor has their own set of adversaries to obstruct, their actions can affect (either favorably or adversely) the other interdictors' interests due to the fact that they all operate on a common network.[1]

Without any communication or coordination between the interdicting agents, one might expect that the cost or effectiveness of the decentralized interdiction strategy is far from optimal (from a centralized point of view). Yet there has been little work so far to study and address the loss in efficiency associated with decentralized decision-making in network interdiction. As multi-agency or multi-national collaborative interdiction missions in network-centric warfare environments have become increasingly common and important, we believe that it is critical from a military perspective to understand the *cause* and *remedies* for such efficiency loss. This project is exactly motivated and designed to fill this knowledge gap. More specifically, the planned and accomplished objectives of this research are as follows, with the detailed account of how such objectives are accomplished provided in the next section.

Objective 1. Establish theoretical foundations for the multi-interdictor network interdiction games (MINGs), including proving the existence and uniqueness of equilibria (or lack thereof), and establishing theoretical bounds for the worst-case efficiency loss.

Objective 2. Establish scalable computational frameworks for the MINGs and use the computational methods to numerically quantify the average-case efficiency loss of the MINGs due to the lack of coordination.

Objective 3. Develop scalable computational methods that can compute Nash equilibria under uncertainty for both MINGs and for more general games.

[1]The network here is broadly termed, including, but not limited to, physical infrastructure networks, financial networks, and cyber networks.

2 Accomplishments

2.1 Multi-Interdictor Network Games (MINGs)

As stated in the previous section, MIGNs appear to be new, to the best of our knowledge. One potential reason for the lack of attention paid to such games may be that such games often involve nondifferentiability, as each interdictor's optimization problem usually entails a max-min type of objective functions. Games involving nondifferentiable functions are generally challenging, in terms of both theoretical analysis of their equilibria and computing an equilibrium. While in some cases (such as in the case of shortest-path interdiction), a smooth formulation (through total unimodularity and duality) is possible, such a reformulation will lead the resulting network game to the class of generalized Nash equilibrium problems (GNEPs), in which both the agents' objective functions as well as their feasible action spaces depend on other agents' actions. Although the conceptual framework of GNEPs can be dated to 1950's, rigorous theoretical and algorithmic treatments of GNEPs only began in recent years. Several techniques have been proposed to solve GNEPs, including penalty-based approaches, variational-inequality-based approaches, Newton's method, projection methods, and relaxation approaches. Most of the above-mentioned works focus on GNEPs with shared constraints; that is, a set of identical constraints appear in each agent's feasible action set. Such games are much easier to deal with, both theoretically and computationally, than non-shared-constraint GNEPs. A typical MING, however, leads to a GNEP with non-shared constraints, and hence, is a challenging subject. *Nevertheless, by investigating the special structures of MINGs and their different forms of equivalent formations, we are able to establish a series results regarding their equilibrium existence, and algorithms that can compute such equilibria.*

2.1.1 Shortest-path multi-interdictor games (SPMINGs)

While a MING is a general class of games, the specific form of game played between an individual interdictor and its adversaries needs to be specified. Generally speaking, there are three specific types of MINGs: the shortest-path multi-interdictor games, the maximum-flow multi-interdictor games, and the minimum-cost-flow multi-interdictor games. Our research effort initially focused on SPMINGs due to their tractability, and later was extended to the other types of games.

Definition and Formulation. Formally, we have a set $\mathcal{F} = \{1, \ldots, F\}$ of interdictors or agents, who operate on a network $G = (V, A)$, where V is the set of nodes and A is the set of arcs. Each agent's actions or decisions correspond to interdicting each arc of the network with varying intensity: the decision variables of agent $f \in \mathcal{F}$ are denoted by $x^f \in X^f \subset \mathbb{R}^{|A|}$, where X^f is an abstract set that constrains agent f's decisions. For any agent $f \in \mathcal{F}$, let x^{-f} denote the collection of all the other agents' decision variables; that is, $x^{-f} = (x^1, \ldots, x^{f-1}, x^{f+1}, \ldots, x^F)$. The network obtained after every agent executes its decisions or interdiction strategies is called the aftermath network. The strategic interaction between the agents occurs due to the fact that the properties of each arc in the aftermath network are affected by the combined decisions of all the agents. In addition to the abstract constraint set X^f, we assume that each agent $f \in \mathcal{F}$ faces a total interdiction budget of b^f. The cost of interdicting an arc is linear in the intensity of interdiction; in particular, agent f's cost of interdicting arc (u, v) by x^f_{uv} units is $c^f_{uv} x^f_{uv}$. Without loss of generality, we assume that $b^f > 0$ and $c^f_{uv} > 0$ for each arc $(u, v) \in A$ and for each agent $f \in \mathcal{F}$. The optimization problem for each agent $f \in \mathcal{F}$ is as follows.

$$
\begin{aligned}
\underset{x^f}{\text{maximize}} \quad & \theta^f(x^f, x^{-f}) \\
\text{subject to} \quad & \sum_{(u,v)\in A} c^f_{uv} x^f_{uv} \le b^f, \\
& x^f \in X^f,
\end{aligned}
\tag{1}
$$

where the objective function θ^f is agent f's *obstruction function*, or measure of how much agent f's adversary has been obstructed. Henceforth, we refer to the game in which each agent $f \in \mathcal{F}$ solves the above optimization problem (1) as a MING. The obstruction function θ^f can capture various types of interdiction problems. Typically θ^f is the (implicit) optimal value function of the adversary's network optimization problem parametrized by the agents' decisions, which usually minimizes flow cost or path length subject to flow conservation, arc capacity and side constraints. For a specific SPMING, each agent $f \in \mathcal{F}$ has a target node $t^f \in V$ that it wishes to protect from an adversary at source node $s^f \in V$ by maximizing the length of the shortest path between the two nodes. The agents achieve this goal by committing some resources (e.g. monetary spending) to increase the individual arc lengths on the network: the decision variable x^f_{uv} represents the contribution of agent $f \in \mathcal{F}$ towards lengthening arc $(u, v) \in A$. The arc length $d_{uv}(x^f, x^{-f})$ of arc $(u, v) \in A$ in the aftermath network depends on the decisions of all the agents. In this work, we considered two types of interdiction. The first type of interdiction is *continuous*: in particular, $X^f := \{x^f \in \mathbb{R}^{|A|} : x^f_{uv} \geq 0 \quad \forall (u, v) \in A\}$. The arc lengths after an interdiction strategy $(x^1, \ldots, x^F)$ has been executed are

$$d_{uv}(x^1, \ldots, x^F) = d^0_{uv} + \sum_{f \in \mathcal{F}} x^f_{uv} \quad \forall (u, v) \in A, \tag{2}$$

where x^f_{uv} captures how much agent f extends the length of arc (u, v). We assume that $d^0_{uv} > 0$ for all $(u, v) \in A$.

The second type of interdiction is *discrete*: in this case, $X^f := \{x^f \in \mathbb{R}^{|A|} : x^f_{uv} \in \{0, 1\} \quad \forall (u, v) \in A\}$ and the arc lengths in the aftermath network are

$$d_{uv}(x^1, \ldots, x^F) = d^0_{uv} + e_{uv} \max_{f \in \mathcal{F}} x^f_{uv} \quad \forall (u, v) \in A, \tag{3}$$

where $e_{uv} \in \mathbb{R}_{\geq 0}$ is the fixed extension of arc (u, v). In other words, the length of an arc is extended by a fixed amount if at least one agent decides to interdict it.

Let $P^f = \{p^f_1, p^f_2, \ldots, p^f_{k^f}\}$ be the set of $s^f - t^f$ paths available to agent $f \in \mathcal{F}$. The length of a path $p \in P^f$ is given by

$$d_p(x^1, \ldots, x^F) = \sum_{(u,v) \in p} d_{uv}(x^1, \ldots, x^F), \tag{4}$$

where $d_{uv}(x^1, \ldots, x^F)$ is as defined in equation (2) for continuous interdiction, and as defined in (3) for the discrete case. The optimization problem for each interdicting agent $f \in \mathcal{F}$ is then:

$$\begin{aligned} \underset{x^f}{\text{maximize}} \quad & \theta^f(x^f, x^{-f}) \equiv \min_{p \in P_f} d_p(x^f, x^{-f}) \\ \text{subject to} \quad & \sum_{(u,v) \in A} c^f_{uv} x^f_{uv} \leq b^f, \\ & x^f \in X^f. \end{aligned} \tag{5}$$

Summary of Equilibrium Properties for SPMINGs. We focused on studying pure-strategy Nash equilibria in this work, as opposed to mixed-strategy Nash equilibria (in which agents' strategies are randomized). More specifically, we attempted to answer two questions: (1) if a pure strategy Nash equilibrium exists; (2) if the answer to (1) is yes, is the equilibrium unique.

Equilibrium Existence. For the equilibria of SPMINGs under continuous interdiction, we have the following result (in Publication [P1] listed in Section 4 of this report).

Proposition 1. *Given that each agent $f \in \mathcal{F}$ solves the problem* (5)*, with $d_p(x^f, x^{-f})$ defined as in* (4) *and* (2)*, and assume that the abstract set X_f in* (5) *is nonempty, convex and compact for each $f \in \mathcal{F}$, the SPMING under continuous interdiction has a pure strategy Nash equilibrium.*

Under discrete interdiction, the existence of a pure-strategy equilibrium is not always guaranteed when different interdictors are competing against different adversaries. We have discovered an example to illustrate this situation in [P1]. If, however, the interdictors' adversaries all share the same source and target of a network (the actual topology of the networks can be different), then the following result is shown.

Proposition 2. *Suppose the source and target for each agent in a SPMING are the same. Let x^* denote the optimal solution of the centralized problem. Then x^* is a pure-strategy to the SPMING under discrete interdiction.*

Equilibrium Uniqueness We have found simple instances of the SPMING that multiple equilibria exist. Hence, one may not expect to establish theoretical results regarding the uniqueness of equilibria for any MINGs. Through the instances of the SPMING with multiple equilibria, we observed that all agents may be worse off in a certain equilibrium than in other equilibria. This, together with the lack of uniqueness of equilibria, motivated us to design decentralized algorithms to compute equilibria of SPMINGs (or of MINGs in general) such that the entire path of how a particular equilibrium is reached can be known. In addition, such algorithms shall be able to find multiple equilibrium (for example, by changing the starting point of running the algorithms). The development of such algorithms is reported in the following.

Computation of Equilibria. In developing algorithms to compute equilibria of SPMINGs, we initially focused on SPMINGs with continuous interdiction. Within such, we first investigated the feasibility of using a centralized approach to solve the SPMING, which was to serve as a benchmark (both in terms of performance and in terms of solution accuracy) for the later developed decentralized algorithms. For SPMINGs with discrete interdiction, no known centralized algorithms exist.

A Centralized Algorithm. The term "centralized" here refers to the computational methods that are designed to directly compute an equilibrium of a game (much like of finding an optimal solution of an optimization problem), while ignoring how such an equilibrium is achieved by the strategic agents. In contrast, the term "decentralized" refers to algorithms that specify how each agent reacts to the observed new information of their rivals' past actions[2], and an equilibrium may be reached by such algorithms through iterative interactions among the agents; that is, such algorithms try to mimic the iterative decision-making process among the agents.

For games with differentiability (and convexity), the prevailing approach for centralized algorithms to compute an equilibrium is to write down the first-order optimality condition (aka the KKT condition) of each agent's optimization problem, and then to stack all the KKT conditions together to form a complementarity problem (CP). Newton-based algorithms have been developed in the past several decades to compute a solution of of a CP (that corresponds to the equilibrium of the game with differentiability and convexity). While the SPMING formulation given in (5) is not differentiable, by writing out the dual of the adversary's shortest-path problem, the SPMING can be reformulated into a game with smooth functions, with each agent solving the following optimization problem.

$$
\begin{aligned}
\underset{x^f,\, y^f}{\text{maximize}} \quad & y^f_{t^f} - y^f_{s^f} \\
\text{subject to} \quad & y^f_v - y^f_u \le d_{uv}(x^f, x^{-f}) \quad \forall (u,v) \in A, \\
& \sum_{(u,v)\in A} c^f_{uv} x^f_{uv} \le b^f, \\
& x^f \in X^f, \\
& y^f_v \ge 0 \quad \forall v \in V,
\end{aligned} \tag{6}
$$

[2]Such algorithms also need to specify exactly what information can be observed by each agent at the time they need to make a decision.

The formulation (6) gives us some insight into the structure of strategic interactions among agents in a SPMING. Note that in formulation (6), the objective function for each agent $f \in \mathcal{F}$ only depends on variables indexed by f (in particular, $y^f_{s^f}$ and $y^f_{t^f}$). However, the constraint set for each agent f is parametrized by other agents' variables x^{-f}, which leads to a *generalized Nash equilibrium problem (GNEP)*.

By writing out the optimality conditions of (6) and stacking such conditions of all the agents, a linear complementarity problem of the general form $0 \leq w \perp q + Mw \geq 0$, denoted as LCP$(q, M)$, can be obtained, where the '$\perp$' sign represents the vector product, $q \in \mathbb{R}^d$ and matrix $M \in \mathbb{R}^{d\times d}$ are from input data, and $w \in \mathbb{R}^d$ is the vector of variables to be found. While pivoting-based algorithms, generally known as the Lemke's method, are well-established to solve LCPs, it is known that the method may terminate on a secondary ray (similar to the well-known simplex method for linear programs terminating on an extreme direction). However, we showed in [P1] that barring from degeneracy, Lemke's method will always find a solution of the LCP derived from a SPMING.

Decentralized Algorithms. In designing decentralized algorithms to solve the SPMINGs, we utilized the best-response-type algorithms, also known as Gauss-Seidel algorithms, in which each agent solves their own optimization problem based on the information of other agents' actions so far (assuming such information is available), and then passes his/her optimal solution to the next agent for him/her to continue the iterative algorithm. While the algorithm idea is simple, it is well known that such "naïve" implementation of a Gauss-Seidel (GS) algorithm may fail to converge to equilibria. However, we were able to show convergence of the unmodified GS algorithm for SPMINGs under *discrete interdiction* in [P1], as summarized below.

Proposition 3. *Suppose that the Gauss-Seidel algorithm is applied to a SPMING with discrete interdiction, and the termination criterion $\|\chi_k - \chi_{k-1}\| \leq \epsilon$ is used with $\epsilon < 1$. If the algorithm terminates at χ_k, then χ_k is an equilibrium to the SPMING.*

Corollary 1. *Consider a SPMING with discrete interdiction with common source-target pairs, and assume that the initial arc lengths d and arc extensions e are integral. Suppose that the Gauss-Seidel algorithm is applied to such a problem, and the termination criteria $\|\chi_k - \chi_{k-1}\| \leq \epsilon$ is used with $\epsilon < 1$. Then the algorithm will terminate finitely at an equilibrium.*

For SPMINGs under *continuous interdiction*, no such convergent results can be achieved for the unmodified GS algorithm. Instead, at each iteration, we let an agent solves the so-called regularized version of his/her optimization problem; that is,

$$\begin{aligned} \underset{\chi^f}{\text{maximize}} \quad & \theta^f(\chi^f, \chi^{-f}) - \tau \left\| \chi^f - \overline{\chi}^f \right\|^2 \\ \text{subject to} \quad & \chi^f \in \Xi^f(\chi^{-f}), \end{aligned} \tag{7}$$

where τ is a positive constant. Here the regularization term is evaluated in relation to a candidate point $\overline{\chi}^f$. Note that the point $\overline{\chi}^f$ and the other agents' decision variables χ^{-f} are fixed when the problem (7) is solved. We refer to the overall modified Guass-Seidel algorithm as the regularized Gauss-Seidel (RGS) procedure. Though we were not able to prove unconditional convergence of RGS algorithm on the SPMINGs, due to the fact that SPMINGs are GNEPs with non-shared constraints, we nevertheless had the following results regarding convergence, as documented in [P1].

Proposition 4. *Let $\{\chi_k\}$ be the sequence generated by applying the RGS algorithm to a SPMING with continuous interdiction, wherein each agent solves* (7)*. Suppose that $\{\chi_k\}$ converges to $\bar{\chi}$. Then $\bar{\chi}$ is an equilibrium to the SPMING.*

Corollary 2. *Consider applying the RGS algorithm to a SPMING with continuous interdiction, with common source-target pairs. Let $\{\chi_k\}$ be the sequence generated by the algorithm. If $\bar{\chi}$ is a cluster point of this sequence, then it is also an equilibrium of the SPMING.*

Numerical Results. We applied the algorithms discussed above to study several instances of SPMINGs. The decentralized algorithms were implemented in MATLAB R2010a with CPLEX v12.2 as the optimization solver. The LCP formulation for a SPMING with continuous interdiction was solved using the MATLAB interface for the complementarity solver PATH. Computational experiments were carried out on a desktop workstation with a quad-core Intel Core i7 processor and 16 GHz of memory running Windows 7.

Among the tested instances, the larger network cases are illustrated by Figure 1, and the corresponding numerical results are provided in Table 1.

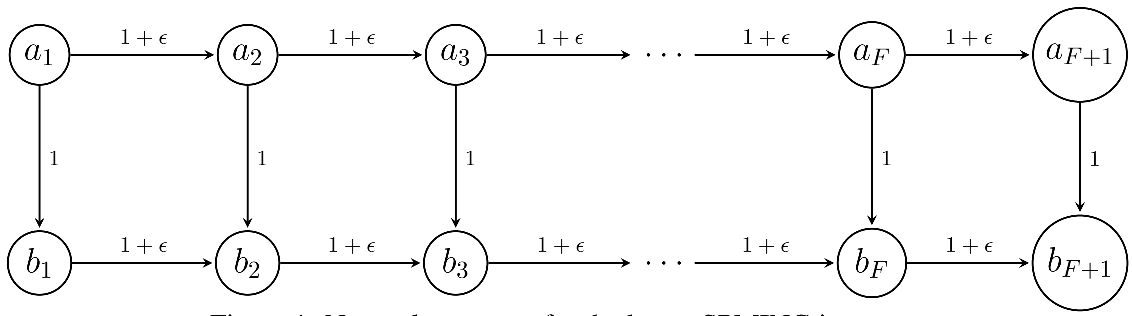

Figure 1: Network structure for the larger SPMING instances.

Table 1: Number of iterations and running times for the network in Figure 1.

	Continuous Interdiction			Discrete Interdiction	
	Decentralized		LCP	Decentralized	
# Agents	# Iters	Runtime (s)	Runtime (s)	# Iters	Runtime (s)
5	3	0.0205	0.0290	5	0.1776
10	5	0.0290	0.1833	3	0.1627
15	11	0.1103	0.7534	3	0.2419
20	5	0.0723	2.1106	3	0.3164
25	13	0.2609	4.8167	3	0.4005
30	15	0.4070	10.2256	3	0.5155
35	10	0.3605	17.7387	3	0.5948
40	41	1.7485	30.2382	3	0.7387
45	12	0.6601	48.6280	3	0.8794
50	12	0.7981	75.0420	3	1.0385

Two key observations from our numerical experiments are: (1) multiple Nash equilibria were indeed found by varying the starting point of the decentralized algorithms; (2) the RGS decentralized algorithm for SPMINGs with continuous interdiction scales very well, compared to the centralized LCP algorithm. It can be seen from Table 1 that with the size of input data increasing, the run time of the LCP algorithm (aka the Lemke's method) increases significantly (which is not a surprise as the Lemke's method is not a polynomial-time algorithm, just like the simplex method); while the run time for the decentralized algorithm remains relatively flat.

Efficiency Loss Quantification. It is well known that a Nash equilibrium outcome may not be optimal from a system perspective (that is, not Pareto optimal) due to the unilateral actions among the strategic agents. This sub-optimality of Nash equilibria, broadly termed as efficiency loss, has been studied in many other game-theoretic settings. Most of such efforts have been focused on routing games, congestion games, and network formation games; while such studies on facility location games, scheduling games, and resource allocation games also appear in the literature. Almost all of the work described above study the worst-case

inefficiency of a given equilibrium concept. Although a few researchers have studied the average inefficiency of equilibria, either theoretically or empirically, and have used it as a basis to design interventions to reduce the inefficiency of equilibria, research in this direction has not received much attention. In this project, we focused on establishing a numerical approach to empirically study the average-case efficiency loss, and comparing it with the worst-case loss. The main purpose is to illustrate that for certain games, even the theoretical worst-case efficiency loss is significant, the measure may be too conservative as the worst case may be caused by pathological instances. In such cases, if the average-case efficiency loss is insignificant, then no additional actions may be needed to improve efficiency, which would save time and money of any central agencies. Motivated by this purpose, we introduced the definition of average case efficiency as follows.

First consider a central planner with a comprehensive view of the network and all the agents' objectives that can pool the agents' interdiction resources and determine an interdiction strategy that maximizes some global measure of how much the agents' adversaries have been obstructed. Let $\boldsymbol{\theta}^{\boldsymbol{c}}(x^1, \ldots, x^F)$ represent the global obstruction function for a given interdiction strategy $(x^1, \ldots, x^F)$. The central planner's problem corresponding to a generic multi-interdictor problem is then:

$$\begin{aligned} \underset{x^1,\, \ldots,\, x^F}{\text{maximize}} \quad & \boldsymbol{\theta}^{\boldsymbol{c}}(x^1, \ldots, x^F) \\ \text{subject to} \quad & \sum_{f \in \mathcal{F}} \sum_{(u,v) \in A} c^f_{uv} x^f_{uv} \leq \sum_{f \in \mathcal{F}} b^f, \\ & x^f \in X^f \quad \forall f \in \mathcal{F}. \end{aligned} \tag{8}$$

(8) is referred to as the centralized problem, and we focused primarily on when the global obstruction function is *utilitarian*; that is,

$$\boldsymbol{\theta}^{\boldsymbol{c}}(x^1, \ldots, x^F) := \sum_{f \in \mathcal{F}} \theta^f(x^f, x^{-f}).$$

A commonly used measure of Nash equilibria's inefficiency is the *price of anarchy*. Formally speaking, let $\mathcal{N}_I$ be the set of all equilibria corresponding to a specific instance I. (In the context of MINGs, an instance consists of the network, obstruction functions, interdiction budgets, and costs). For the same instance I, let $(x^{1^*}, \ldots, x^{F^*})$ denote a globally optimal solution to the centralized problem (8). Then the price of anarchy of the instance I is defined as

$$p(I) := \max_{(x^1_N, \ldots, x^F_N) \in \mathcal{N}_I} \frac{\boldsymbol{\theta}^{\boldsymbol{c}}(x^{1^*}, \ldots x^{F^*})}{\boldsymbol{\theta}^{\boldsymbol{c}}(x^1_N, \ldots, x^F_N)}. \tag{9}$$

Let $\mathcal{I}$ be the set of all instances of a game.[3] If the worst equilibrium has a zero objective value while the global optimal value of the centralized problem is nonzero, p is set to be infinity. In addition to the price of anarchy for an instance of a game, we also define the worst-case price of anarchy over all instances of the game (denoted as $w.p.o.a$) as

$$w.p.o.a := \sup_{I \in \mathcal{I}} p(I). \tag{10}$$

In addition to being too conservative, the measure $w.p.o.a$ has another drawback that explicit theoretical bounds on the worst-case price of anarchy may be difficult to obtain for general classes of games. In fact most of the related research has focused on identifying classes of games where such bounds may be derived. In this work, we showed how our proposed decentralized algorithms can be used to empirically study the

[3]It is assumed implicitly that for all $I \in \mathcal{I}$, the set $\mathcal{N}_I$ is nonempty and a global optimal solution to the centralized problem exists. By convention, p is set to 1 if the worst equilibrium as well as the global optimal solution to the centralized problem both have zero objective value.

average-case efficiency loss (denoted by $a.e.l$). Let $\mathcal{I}'$ denote a finite set such that $\mathcal{I}' \subset \mathcal{I}$, and let $|\mathcal{I}'|$ denote the cardinality of the the set $\mathcal{I}'$. Then

$$a.e.l(\mathcal{I}') := \frac{1}{|\mathcal{I}'|} \sum_{I \in \mathcal{I}'} p(I). \tag{11}$$

In other words, the average-case efficiency loss is the average value of $p(I)$ as defined in (2) over a set of sampled instances $\mathcal{I}' \subset \mathcal{I}$ of a game.

To numerically study the average-case efficiency loss of a SPMING, we first established a theoretical lower bound on the worst-case price of anarchy of SPMINGs, which is $(F+1)/(2+\epsilon)$, with F being the number of interdicting agents.

Using the RGS decentralized algorithm to find multiple equilibria and sampling from a range of input data based on the network structure in Figure 1, we were able to compute the average-case efficiency loss. The comparison between such loss and the lower-bound of the worst-case price of anarchy is provided in Figure 2. It can be observed that for the particular graph structure under consideration, we observe that the average efficiency loss grows at a much lower rate than the worst-case efficiency loss.

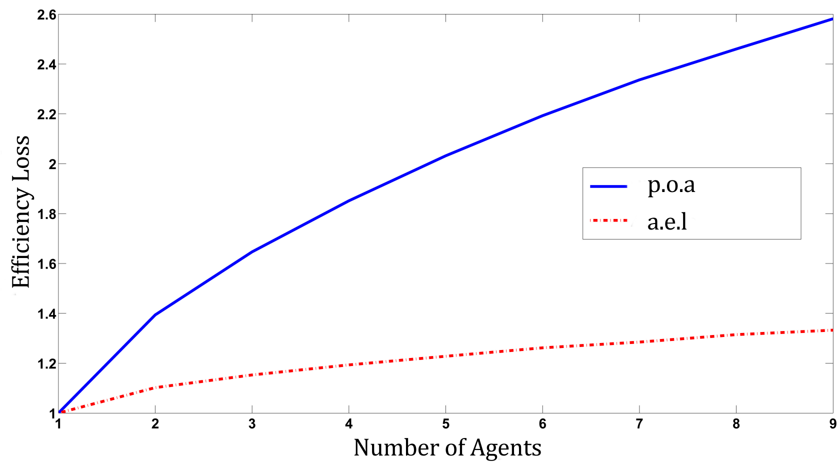

Figure 2: Efficiency loss with respect to the number of interdicting agents.

2.1.2 Maximum-flow multi-interdictor games (MFMINGs)

Built upon our experiences with the SPMINGs, we extended our work to study another type of MING: maximum-flow multi-interdictor network games (MFMINGs). Such game models can be used to describe situations in which adversaries wish to maximize the flow of some goods between two nodes on a network. Each interdictor's aim is to minimize this maximum flow by means of taking actions to reduce arc capacities on the network.

Formally, let the network on which the various agents act be described by the digraph $G = (V, A)$. Each arc $(i, j) \in A$ has a capacity u_{ij}. For agent $f \in \mathcal{F}$, the adversary wishes to maximize the flow between source node s^f and target node t^f. Agent f then wishes to minimize the maximum flow $s^f - t^f$ flow. Each

agent f then solves the following optimization problem:

$$\begin{array}{ll} \underset{x^f}{\text{minimize}} & \theta^f(x^f, x^{-f}) = \left(\begin{array}{ll} \max\limits_{y} & y^f_{t^f s^f} \\ \text{s.t.} & \sum\limits_{v \in V} y^f_{uv} - \sum\limits_{v \in V} y^f_{vu} = 0, \ \forall u \in V \\ & y^f_a \le u_a(x^f, x^{-f}), \ \forall a \in A \backslash \{(t^f, s^f)\} \end{array} \right) \\ \text{subject to} & \sum\limits_{(u,v) \in A} c^f_{uv} x^f_{uv} \le b^f \\ & x^f \in X^f. \end{array} \tag{12}$$

The x variables represent the actions of the interdicting agent, with each agent f's interdiction action subject to a budget constraint, along with other possible constraints summarized in the abstract set X^f. The y variables represent the decisions of the adversaries. That is, y_{uv} is the flow on arc (u, v) on the aftermath network.

As in the SPMING, the effect of interdiction decisions (x^f, x^{-f}) on the arc capacities depend on the type of interdiction. If interdiction is assumed to be continuous and additive, then the relationship is given as follows:

$$u_{ij}(x^f, x^{-f}) = \max \left\{ (u^0_{ij} - \sum_{f=1}^{F} x^f_{ij}), 0 \right\},$$

where x^f is restricted to be component-wise non-negative. If on the other hand interdiction is binary, the arc capacities are given by

$$u_{ij}(x^f, x^{-f}) = u^0_{ij}(1 - \max_{f \in \{1,\ldots,F\}} x^f_{ij}).$$

While under either the continuous or discrete interdiction, the objective function in (12) is not smooth. Utilizing LP duality and by introducing an auxiliary variable (τ), we obtained a smooth reformulation for the MFMING under continuous interdiction. (For discrete interdiction, other than the integer variables, everything else is smooth in the reformulation.)

$$\begin{array}{ll} \underset{x^f,\, \beta^f,\, \alpha^f,\, \tau}{\text{minimize}} & \sum\limits_{(i,j) \in A} \beta^f_{ij} \tau \\ \text{subject to} & \tau \ge u^0_{ij} - \sum\limits_{h=1}^{F} x^h_{ij} \\ & \tau \ge 0 \\ & \beta^f_{ij} + \alpha^f_i - \alpha^f_j \ge 0 \quad \forall (i,j) \in A \\ & \alpha^f_{t^f} - \alpha^f_{s^f} \ge 1 \\ & \beta^f_{ij} \ge 0, \quad \forall (i,j) \in A \\ & \sum\limits_{(u,v) \in A} c^f_{uv} x^f_{uv} \le b^f \\ & x_f \in X_f. \end{array} \tag{13}$$

When each interdicting agent solves (13), the overall MFMING leads to a GNEP, as other agents' decision variables enter into the first constraint: $\tau \ge u^0_{ij} - \sum_{h=1}^{F} x^h_{ij}$. The main difficulty of such a game lies in the fact that the objective function in (13) involves a bilinear term, making the MFMING an instance of nonconvex games.

Nonconvex games (with differentiability) in general are very difficult to analyze in terms of equilibrium properties due to the fact that the first-order optimality conditions (FOCs) are only necessary conditions for each agent's optimization problem. In addition, the complementarity problems derived from stacking all agents' FOCs usually lack of desired properties (mainly monotonicity of set-valued mappings) to deduce any results. Nonconvex games with integer variables [4] are even harder to analyze, with few known results in the literature. Nevertheless, we were able to show equilibria existence for a special class of MFMINGs. (See [D1] for details.)

Proposition 5. *For a MFMING under either continuous or discrete interdiction, an equilibrium always exists if the source and target nodes are the same for each agent, i.e.* $s^f = s$ *and* $t^f = t$ *for* $f = 1, \ldots, F$.

Other theoretical properties of MFMING equilibria, as well as computational approaches, are the subjects of the ongoing research of PIs and their collaborator, Professor Jong-Shi Pang at University of Southern California. We are working on to develop more general theoretical and computational frameworks for broader classes of nonconvex games. Should it be successful, it would represent a major step forward in the field of computational game theory.

2.2 Cost Sharing and Network Fortification

As a side project inspired by the primary research activities funded by this grant, we studied stochastic linear programming games: a class of stochastic cooperative games whose payoffs under any realization of uncertainty are determined by a specially structured linear program. These games can model a variety of settings, including a scenario in which players can cooperate to fortify a network against an impending attack or natural disaster with uncertain effects on the network. We focused on the core of these games under an allocation scheme that determines how payoffs are distributed before the uncertainty is realized, and allows for arbitrarily different distributions for each realization of the uncertainty. Assuming that each player's preferences over random payoffs are represented by a concave monetary utility functional, we proved that these games have a nonempty core. Furthermore, by establishing a connection between stochastic linear programming games, linear programming games and linear semi-infinite programming games, we showed that an allocation in the core can be computed efficiently under some circumstances. The major results of this line of research are documented in [P2].

2.3 Bridging Game Theory and Large-Scale Optimization under Uncertainty

2.3.1 Potential games and their relationship to optimization

In the above-described work, all input data were assumed to be deterministic. In reality, however, uncertainties are present in various aspects of a decision-making process, ranging from uncertain input data to incomplete information of rivals' actions, in a game setting. To extend our work, we investigated into games with exogenously uncertain input data (namely, the uncertainty of the input data are uncontrollable by the agents).[5]

For a generic games under exogenous uncertainty, each agent f ($f = 1, 2, \ldots, F$) tries to solve the following optimization problem

$$\begin{array}{ll} \underset{x_f}{\text{minimize}} & \phi_f(x) = \mathbb{E}\left[\theta_f(x_f, x_{-f}; \xi)\right] \\ \text{subject to} & x_f \in X_f. \end{array} \tag{14}$$

[4] Here the nonconvexity refers to the fact that even not considering the integrality, the remaining game is still nonconvex.

[5] A related but different concepts of games under uncertainty is games of incomplete information. In such games, the uncertainty is mainly due to the incomplete knowledge of the agents towards their rivals in terms of the rivals' utility functions and/or feasible action spaces. Such incomplete information may or may not be caused by exogenous uncertainties uncontrollable by the agents. In such games, the pertinent equilibrium concept is Bayesian Nash equilibrium (BNE). Developing algorithms to compute BNE is the ongoing research effort of PI Liu.

By using the expected value as the objective function, we assumed that the agents are risk-neutral. We denote the game above as NEP$(\phi_f, X_f)_{f=1}^F$. A set of player decision variables $x^* = (x_1^*, \ldots, x_F^*)^T \in \mathcal{X} := \prod_{f=1}^F X_f$ is a Nash equilibrium to NEP$(\phi_f, X_f)_{f=1}^F$ if and only if the following condition holds:

$$\phi_f(x_f^*, x_{-f}^*) \leq \phi_f(y_f, x_{-f}^*) \quad \forall y_f \in X_f. \tag{15}$$

Since the above definition is not different than that to a deterministic game, all the theories regarding equilibrium existence and uniqueness are applicable to the game NEP$(\phi_f, X_f)_{f=1}^F$ as well. On the computation side, algorithms for computing an equilibrium of NEP$(\phi_f, X_f)_{f=1}^F$ have been well studied. Most of such algorithms rely on converting the game into a (finite-dimensional) variational inequality or a complementarity problem through stacking the first-order optimality conditions of (14). Such algorithms are termed as the centralized algorithms. In this research effort, we were interested in designing decentralized algorithms to compute an equilibrium, which can achieve two things that centralized algorithms cannot achieve: (1) the entire path of how an equilibrium is achieved can be obtained through decentralized algorithms, which will provide more insights on how agents interact with each other in a game, and on why certain equilibria are preferred than the others; (2) multiple equilibria may be found through decentralized algorithms.[6] As a result, the average-case efficiency loss of noncooperated equilibria can then be studied using the computational-based approach established in this research, as described above.

While the conceptual ideas of decentralized algorithms (broadly termed that include any type of learning-based algorithms, as well as the best-response-type algorithms) for games are generally easy, and implementation is usually straightforward, the convergence (to an equilibrium), however, is very difficult to establish in general. As a result, we focused on a specific class of games, known as the exact potential games.

Definition 1. [7] *The NEP$(\phi_f, X_f)_{f=1}^F$ is said to have an* exact potential function $\bar{P} : \mathbb{R}^n \to \mathbb{R}$ *if the following holds for $f = 1, \ldots, F$:*

$$\begin{aligned} \phi_f(x_f, x_{-f}) - \phi_f(y_f, x_{-f}) \;&=\; \bar{P}(x_f, x_{-f}) - \bar{P}(y_f, x_{-f}) \\ &\qquad \forall x_f, y_f \in X_f, \text{ and } x_{-f} \in \prod_{\substack{f' \in \mathcal{F} \\ f' \neq f}} X_{f'}. \end{aligned}$$

A game that has an exact potential function is called an exact potential game.

A key feature of potential games is that under certain conditions, many decentralized algorithm have been show to be convergent to an equilibrium. Such a result, however, has not been established (to the best of our knowledge) for potential games under uncertainty. One of the major contributions of this line of research is to fill in such a void.

While the class of exact potential games may be restrictive, it nonetheless includes several very important games, such as the routing games and Nash-Cournot games. The the SPMINGs and MFMINGs described above with their dual formulations ((6) and (13), respectively) are both instances of exact potential games as well. This was exactly why we chose to focus on potential games in this research effort. In addition, based on the relationship between a potential game and a single optimization problem, the decentralized algorithms we developed are also applicable to solve large-scale optimization problems with block-structure. Such a relationship is formally stated in [P3] and is presented below.

Proposition 6. *Suppose that NEP$(\theta_f, X_f)_{f=1}^F$ is a player-wise convex game with an exact potential function $\bar{P}$, and each $\theta_f(\cdot, x_{-f}; \xi)$ is continuously differentiable for any $x_{-f} \in X_{-f}$ and for any $\xi \in \Xi$. Then x^**

[6] There are centralized algorithms that are designed to compute all equilibria of a game, such as the works done by Professor Karl Shcmedders in Northwestern University and his collaborators. However, the class of the games that such algorithms are applicable is very restricted, and is very different than our interests, which stemmed from the multi-interdictor network games.

[7] The defition is due to Dov Monderer and Lloyd S Shapley. "Potential games." *Games and Economic behavior*, 14(1):124–143, 1996.

is an equilibrium to the NEP$(\theta_f, X_f)_{f=1}^F$ *if and only if* $x^* \in X$ *is a stationary solution of the following problem:*

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \bar{P}(x) \\ \textit{subject to} & x \in X \equiv \displaystyle\prod_{f=1}^{F} X_f. \end{array} \tag{16}$$

Based on the above proposition, it can be seen that finding a stationary point of the following optimization problem is equivalent to a game:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \mathbb{E}\left[P(x_1, x_2, \ldots, x_F; \xi)\right] \\ \text{subject to} & x = (x_1, x_2, \ldots, x_F) \in X \equiv X_1 \times X_2 \times \cdots \times X_F, \end{array} \tag{17}$$

where $X_f \in \Re^n_f$ for each $f = 1, \ldots, F$; namely, the feasible region X in (17) is the Cartesian product of the sets X_f's. This problem can be viewed as a game in which each agent controls their own decision variable x_f, while the objective function of each agent is the same, which is the objective function in (17). This is easily seen to be a exact potential game since the common objective function is the exact potential function.

2.3.2 Decentralized algorithms

In this work, we also focused on best-response (BR)-type decentralized algorithms (as opposed to other learning based algorithms, such as fictitious play, regret matching, etc). In addition to the Gauss-Seidel-type implementation of the BR algorithm, there is another way of implementation, generally referred to as Gauss-Jacobi. Its essence is to have all the agents update their best responses all at once, based on the previous iterations' information (as opposed to in Gauss-Seidel approach where the agents take turns to update their best-response). The computational advantage of Gauss-Jacobi's approach is apparent as it would allow for natural parallel computing. The disadvantage, however, is that Gauss-Jacobi-type algorithms can rarely be shown to be convergent to an equilibrium (or to an optimal solution) (making them only heuristic algorithms), even if Gauss-Seidel-type algorithms can be shown to be convergent when applied to the same problems. In this work, however, we indeed can rigorously prove the convergence to an equilibrium of both the Gauss-Seidel and Gauss-Jacobi type algorithms, given that we add a regularization term to the objective function.

Having the convergence of the BR-type decentralized algorithms is not enough, however, in our case, as we still have uncertainties to deal with. A common approach (as in the stochastic programming community) is to sample from the uncertain quantities, and then solve the sampled problem as an approximation. Such a general approach is termed as the sample average approximation (SAA) method, and theories on the asymptotic convergence to the original problem's optimal solution with increasing the sample sizes have been well established. However, our problem is a game with a fixed sample of the input data. While it is a natural idea that an equilibrium of the original game under uncertainty can be reached through finding an equilibrium of the sampled game (which is a deterministic game) and by increasing the sample size, the proof however would really on the epiconvergence theory, instead of the typical techniques used in the SAA method's convergence. While we provided detailed proof of such convergence in [P3], a side benefit of the proof is that it weakens the conditions required when applying the general SAA approach to the block-structured stochastic programming problem 17. As the conditions known in the literature require the objective functions in 17 to be uniformly convergent with increasing sample sizes, we showed that only epiconvergence of the objective function is needed.

The two modified BR-type algorithms are to be provided below, followed by their convergence results. Notation-wise, we use $\hat{\phi}_f(x)$ to denote the sample averaged objective function of an agent f with a fixed sample size N; that is, with a fixed sample of size N, each agent $f = 1, \ldots, F$ solves the following

optimization problem:

$$\begin{aligned} \underset{x_f}{\text{minimize}} \quad & \hat{\phi}_f(x) = \frac{1}{N}\sum_{j=1}^{N}[\theta_f(x_f, x_{-f}, \xi_j)] \\ \text{subject to} \quad & x_f \in X_f. \end{aligned}$$

We refer to the (deterministic) game with each agent solving the above problem as SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$.

Algorithm 1 Regularized Gauss-Seidel (RGS) algorithm

Step 0: Initialize - Set $x^0 \leftarrow (x_f^0)_{f=1}^F$, $k \leftarrow 0$.
Step 1: Termination Check: **IF** x^k satisfies termination criteria, **THEN STOP**
Step 2: Main Iteration:
FOR $f = 1, \ldots, F$, let x_f^{k+1} solve

$$\begin{aligned} \underset{x_f}{\text{minimize}} \quad & \hat{\phi}_f(x_1^{k+1}, \ldots, x_{f-1}^{k+1}, x_f, x_{f+1}^k, \ldots, x_F^k) + \tau \left\| x_f - x_f^k \right\|^2 \\ \text{subject to} \quad & x_f \in X_f. \end{aligned} \tag{18}$$

Step 3: Update: $x^{k+1} = (x_f^{k+1})_{f=1}^F$.

Theorem 1. *(Convergence of Algorithm 1, [P3]) Suppose NEP$(\theta_f(\cdot, \xi), X_f)_{f=1}^F$ is an exact potential game and is player-wise convex for each $\xi \in \Omega$. In addition, each $\theta_f(\cdot, x_{-f}; \xi)$ is continuously differentiable for any $x_{-f} \in X_{-f}$ and for any $\xi \in \Xi$. Then the following statements hold true:*

1. *Every limit point $\hat{x}$ of the sequence x^k generated by Algorithm 1 is an equilibrium to SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$.*
2. *Suppose further that the family of approximate functions $\{\hat{\phi}_f\}_{f=1}^F$ multi-epiconverges to $\{\phi_f\}_{f=1}^F$ on the set $\mathcal{X}$. In this case, if $\hat{x} \to x^*$ as $N \to \infty$, then x^* is an equilibrium of the original game NEP$(\theta_f, X_f)_{f=1}^F$.*

Algorithm 2 Regularized Gauss-Jacobi (RGJ) Algorithm for SNEP$(\theta_f, X_f)_{f=1}^F$

Step 0: Initialize - Set $N > 0$, $\tau > 0$, $x^0 \leftarrow (x_f^0)_{f=1}^F \in \mathcal{X}$, $k \leftarrow 0$.
Step 1: Termination Check: **IF** x^k satisfies termination criteria, **THEN STOP**
Step 2: Main Iteration:
FOR $f = 1, \ldots, F$, let x_f^{k+1} solve

$$\begin{aligned} \underset{x_f}{\text{minimize}} \quad & \hat{\phi}_f(x_f, x_{-f}^k) + \tau \left\| x_f - x_f^k \right\|^2 \\ \text{subject to} \quad & x_f \in X_f. \end{aligned} \tag{19}$$

Step 3: Update: $x^{k+1} = (x_f^{k+1})_{f=1}^F$.

Theorem 2. *(Convergence of Algorithm 2, [P3]) Suppose the NEP$(\theta_f(\cdot, \xi), X_f)_{f=1}^F$ satisfies all the conditions in Theorem 1. In addition, for each $f = 1, \ldots, F$, $\nabla_x \phi^f(\cdot, \xi)$ is Lipschitz continuous for each $\xi \in \Omega$, and τ is chosen such that*

$$2\, c_\tau = \min_{f \in \mathcal{F}} \inf_{x \in \mathcal{X}} c_{\tau_f}(x) \geq L_{\nabla P}, \tag{20}$$

where c_{τ_f} is the coefficient of strong convexity of $\hat{P}(x_f, x_{-f}^k) + \tau \left\| x_f - x_f^k \right\|^2$. Then the following statements hold true:

1. *Every limit point $\hat{x}$ of the sequence x^k generated by Algorithm 2 is an equilibrium to SAANEP$(\hat{\phi}_f, X_f)_{f=1}^F$.*

2. *Suppose further that the family of approximate functions $\{\hat{\phi}_f\}_{f=1}^F$ multi-epiconverges to $\{\phi_f\}_{f=1}^F$ on the set $\mathcal{X}$. In this case, if $\hat{x} \to x^*$ as $N \to \infty$, then x^* solves SNEP$(\theta_f, X_f)_{f=1}^F$.*

Numerical Implementation. We applied both the RGS and RGJ algorithms to various instances of two classes of exact potential games: Nash-Cournot games and self-routing games. The detailed accounts of the game setup and input data are provided in [P3] and [D1]. Several important observations from the numerical experiments are as follows: (1) multiple equilibria indeed were found when applying the RGS algorithm to self routing games; (2) the performance improvement (in terms of convergence time) with parallel implementation of the RGJ algorithm is close to be linear scaling; that is, let M denote the number of CPUs used to carry out each iteration in Algorithm 2; then the overall convergence time of the RGJ algorithm is approximately T/M, where T is the run time when $M = 1$; (3) both the number of iterations and the overall convergence time are sensitive to the value of the regularization parameter τ, with the bigger the τ is, the worse the overall performance is.

3 Personnel Supported

- Andrew L. Liu (PI), Assistant Professor: salary support for some of the summer months from 2013 to 2015 and conference travel support.

- Nelson U. Uhan (Co-PI), Assistant Professor: summer salary for the past 3 years as well as travel support to conferences.

- Harikrishnan Sreekumaran (graduate student): fully supported by this AFOSR grant from September 30, 2012 to September 2015. He contributed substantially to nearly all the above-mentioned accomplishments (except for 2.2), which consisted of his dissertation. He successfully defended his dissertation on July 23, 2015.

- Run Chen (graduate student): partially supported by this AFOSR grant during some months from 2013 to 2015. Her works related to this project include the development of decentralized algorithms for finding a Bayesian Nash equilibrium for games under incomplete information. Such works are documented in [P5].

- Zibo Zhao (graduate student): partially supported by this AFOSR grant during some months from 2013 to 2015. His works related to this project include designing and developing an agent-based simulator in Matlab to study the evolution of the games in which each agent makes their decisions based on some commonly observed information, such as the market price of certain goods that the agents need to purchase.

4 Technical Publications, Conference Presentations and Seminars

4.1 Doctoral Dissertation

[D1] H. Sreekumaran. "Decentralized Algorithms for Nash Equilibrium Problems – Applications to Multi-Agent Network Interdiction Games and Beyond." Ph.D. Dissertation. School of Industrial Engineering, Purdue University, October, 2015.

[D2] R. Chen. "Decentralized Algorithms and Their Applications in Power Systems." School of Industrial Engineering, Purdue University, Expected in 2017.

4.2 Journal Publications

[P1] H. Sreekumaran, A. R. Hota, A. L. Liu, N. A. Uhan and S. Sundaram. Multi-Agent Decentralized Network Interdiction Games. To be submitted to *European Journal of Operations Research*; preprint available at `http://arxiv.org/abs/1503.01100`.

[P2] N. A. Uhan. Stochastic linear programming games with concave preferences. *European Journal of Operational Research*, 243(2) 637-646, 2015.

[P3] H. Sreekumaran and A. L. Liu. Bridging Game Theory and Large-Scale Optimization: The Case of Potential Games under Uncertainty and Expected-Value Optimization. In preparation. To be submitted to *SIAM Journal on Optimization*.

[P4] H. Sreekumaran and A. L. Liu. Equilibrium Wind Hedge Contract Structures through Nash Bargaining. To be submitted to *Operations Research*.

[P5] R. Chen, H. Sreekumaran, and A. L. Liu. Decentralized Algorithms to Compute Nash Equilibria under Uncertainty: The Case of Nash-Cournot Games. In preparation. To be submitted to *Operations Research*.

4.3 Conference Presentations and Seminars

[CS1] A. L. Liu. On potential games and generalized Nash equilibrium problems. Workshop on Complementarity and its Extensions. Singapore. December 2012.

[CS2] H. Sreekumaran. Multi-agent network interdiction games. INFORMS Annual Meeting. Minneapolis, MN. October 2013.

[CS3] H. Sreekumaran. Potential games with exogenous uncertainty. INFORMS Annual Meeting. San Francisco, CA. November 2014.

[CS4] A. L. Liu. Convergent decentralized algorithms for certain Nash equilibrium problems. Seminar, Daniel J. Epstein Department of Industrial and Systems Engineering, University of Southern California, September 15, 2015.

[CS5] H. Sreekumaran. Distributed algorithms for games under exogenous uncertainty. 22nd International Symposium on Mathematical Programming, Pittsburgh, PA. July 2015.

[CS6] A. L. Liu. Decentralized algorithms for block-structured stochastic programs and potential games under uncertainty. INFORMS Annual Meeting, Philadelphia, PA. November 2015.

[CS7] A. L. Liu. Convergent decentralized algorithms for certain Nash equilibrium problems. Seminar, Department of Industrial and Systems Engineering, Lehigh University, December 2, 2015.

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Final Report

Primary Contact E-mail
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765-494-4763

Organization / Institution name

Purdue University

Grant/Contract Title
The full title of the funded effort.

Decentralized Network Interdiction Games

Grant/Contract Number
AFOSR assigned control number. It must begin with "FA9550" or "F49620" or "FA2386".

FA9550-12-1-0275

Principal Investigator Name
The full name of the principal investigator on the grant or contract.

Andrew L. Liu

Program Manager
The AFOSR Program Manager currently assigned to the award

Fariba Fahroo

Reporting Period Start Date

09/30/2012

Reporting Period End Date

09/29/2015

Abstract

This project established the theoretical and computational foundation for a new class of games, termed as the multi-interdictor network games (MINGs). Such games naturally arise in the context of various military and security applications in which multiple interdictors with differing objectives operating on a common network. Within the general class of MINGs, we proved existence of Nash equilibria for specific subclasses of such games, including the shortest-path and maximum-flow multi-interdictor network games, and developed scalable algorithms to compute such equilibria. In addition, we provided theoretical bounds on the worst-case efficiency loss of equilibria for the shortest-path games, with such loss caused by the lack of coordination among noncooperative interdictors, and prosed an empirical approach utilizing decentralized algorithms to study the average-case efficiency loss. Finally, we developed convergent decentralized algorithms based on the connection between the class of potential games (to which the MINGs belong) and optimization that will advance the computation of

large-scale optimization problems.

In summary, this research project has led to the following three accomplishments. 1. Established theoretical foundations for MINGs, including proving the existence and uniqueness of equilibria (or lack thereof), and establishing theoretical bounds for the worst-case efficiency loss. 2. Established scalable computational frameworks for the MINGs and use the computational methods to numerically quantify the average-case efficiency loss of the MINGs due to the lack of coordination. 3. Developed scalable computational methods that can compute Nash equilibria under uncertainty for both MINGs and for more general games.

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Archival Publications (published) during reporting period:

H. Sreekumaran, A. R. Hota, A. L. Liu, N. A. Uhan and S. Sundaram. Multi-Agent Decentralized
Network Interdiction Games. Preprint
available at http://arxiv.org/abs/1503.01100.

N. A. Uhan. Stochastic linear programming games with concave preferences. European Journal of Operational Research, 243(2) 637-646, 2015.

Changes in research objectives (if any):

Change in AFOSR Program Manager, if any:

Extensions granted or milestones slipped, if any:

AFOSR LRIR Number

LRIR Title

Reporting Period

Laboratory Task Manager

Program Officer

Research Objectives

Technical Summary

Funding Summary by Cost Category (by FY, $K)

	Starting FY	FY+1	FY+2
Salary			
Equipment/Facilities			
Supplies			
Total			

Report Document

Report Document - Text Analysis

Report Document - Text Analysis

Appendix Documents

2. Thank You

E-mail user

Jan 08, 2016 15:39:38 Success: Email Sent to: andrewliu@purdue.edu

Response ID: 5645

Survey Submitted:	Jan 8, 2016 3:39 PM
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User Agent:	Mozilla/5.0 (Macintosh; Intel Mac OS X 10_10_1) AppleWebKit/537.36 (KHTML, like Gecko) Chrome/47.0.2526.106 Safari/537.36
Http Referrer:	http://www.wpafb.af.mil/library/factsheets/factsheet.asp?id=9389
Page Path:	1 : (SKU: 1) 2 : Thank You (SKU: 2)
SessionID:	1452284888_56901bd8e83f01.06148294

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